Electromagnetic Field Theory

Books:

Electrical

1. Sadiku, Matthew N, “Elements of Electromagnetics”, Oxford University Press, ISBN: 0195103688, Latest Edition.
2. William Hayt and John A. Buck, “Engineering Electromagnetics”, McGraw-Hill,ISBN: 0073104639, Latest Edition.
3. Kong J. A., “Electromagnetic Wave Theory”, Cambridge, Latest Edition.
4. John D. Kraus, “Engineering Electromagnetics”, McGraw-Hill Inc., New York, Latest Edition
5. N. N. Rao, “Elements of Engineering Electromagnetics”, Pearson Education, Latest Edition

Electronics

1. Electromagnetic waves & radio system by Jorden R.F.
2. Principle and applications of Electromagnetic fields by Ptonsey R and Collin R.P
3. Applied Electromagnetic by Planus M.A.

Fundamentals of Applied Electromagnetics by Fawaz T. Ulaby et. al. latest edition

Distribution of Marks

▪ Assignments	= 12.5 marks
▪ Quizzes	= 12.5 marks
▪ Mid term exam	= 25 marks
▪ Final term exam	= 50 marks

Introduction

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What is Electromagnetic Field Theory ?

- Electromagnetics or Electromagnetism is about the combination of electricity and magnetism
- We know the relation between electricity and magnetism
- Field is any area. In this case, field is the area where the combined effect of electricity and magnetism can be felt
- Theory is simply defined as a system of ideas intended to explain something

Theories of Electromagnetic Field in Real Life ?

- Wireless communication (Antennas)
- RADAR
- Machines and Drives
- Biomedical applications
- Sensors

Short Review of Some Fundamentals

Fundamental SI Units

Dimension	Unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric Current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol

- Derived Units

Prefixes to Represent Numbers/Units

Prefix	Symbol	Magnitude
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}

Scalar and Vector Quantities

- Scalar Quantity
  - Physical quantities requiring magnitude for complete description
  - Represented by medium-weighted italic font
- Vector Quantity
  - Physical quantities requiring magnitude and direction for complete description
  - Represented by bold face font (or a normal letter with arrow above it)
  - Magnitude is represented by a medium-weighted italic font
  - Direction is represented by a unit vector (bold face letter with circumflex above it)

$$\mathbf{E} = E\hat{\boldsymbol{x}}$$

The Nature of Electromagnetism

Nature of Electromagnetism

- Electromagnetic force is one of the natures fundamental forces
- It operates at the atomic scale
- Its effects can be transmitted through electromagnetic waves through free space and material media
- Source of electric and magnetic fields
- Combination of electric and magnetic fields (electromagnetic field)

Electric Field

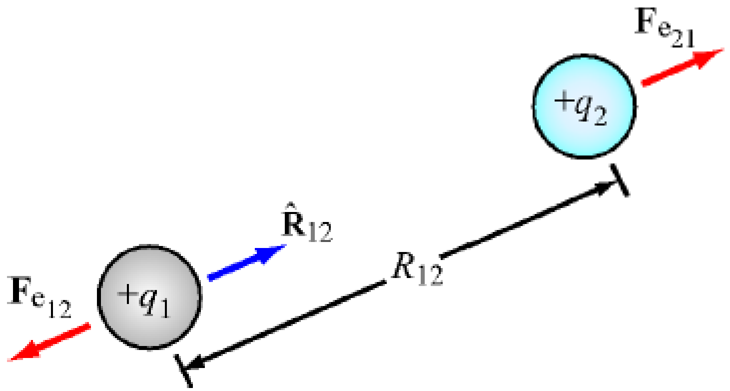
Electric Field

- Source of electric field is electric charge
- Electric charge may have positive or negative polarity
- The resulting force may be attractive or repulsive
- All matter is composed of neutrons (neutral), protons (positively charged), and electrons (negatively charged)
- The fundamental quantity of charge is that of a single electron denoted by e

$$e = 1.6x10^{-19} Coulomb$$

- Charge of a single electron is $q_e = -e$, and that of a proton is $q_p = e$

Coulomb's Law



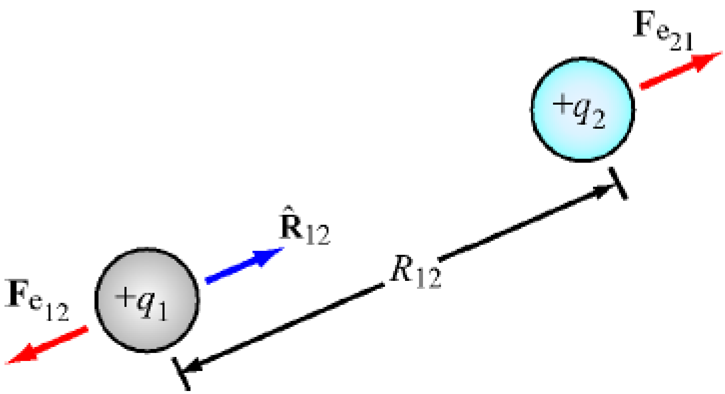
Coulomb's experiments demonstrated that:

(1) *two like charges repel one another, whereas two charges of opposite polarity attract,*

(2) *the force acts along the line joining the charges, and*

(3) *its strength is proportional to the product of the magnitudes of the two charges and inversely proportional to the square of the distance between them.*

Coulomb's Law

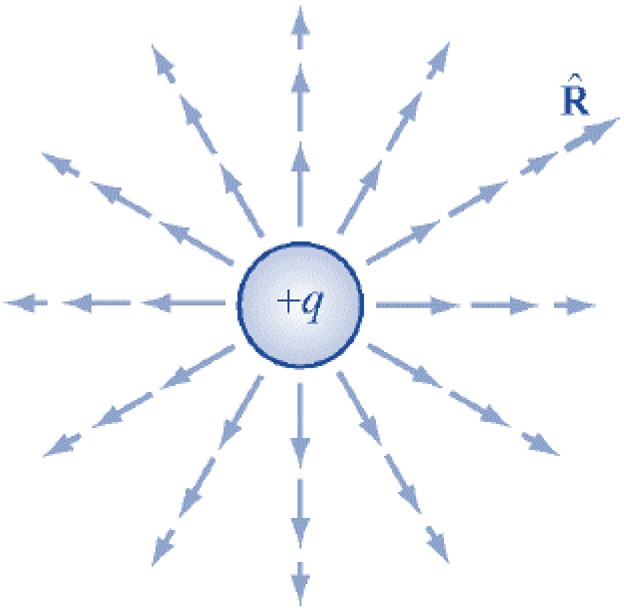


$$\mathbf{F}_{e_{21}} = \hat{\mathbf{R}}_{12} \frac{q_1 q_2}{4\pi \varepsilon_0 R_{12}^2} \quad \text{(N)} \quad \text{(in free space)}$$

- Where F_{e21} is the electric force acting on charge $q2$ due to charge $q1$
- R_{12} is the distance between the two charges
- $\widehat{R_{12}}$ is the unit vector pointing from $q1$ to $q2$
- ε_0 is a universal constant called the electrical permittivity of free space

$$\varepsilon_0 = 8.854 \text{ x } 10^{-12} F/m$$

- $F_{e12} = -F_{e21}$



$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon_0 R^2} \quad \text{(V/m)} \quad \text{(in free space)},$$

If any point charge q' is present in an electric field $\mathbf{E}$ *(due to other charges), the point charge will experience a force acting on it equal to* $\mathbf{F}_e = q'\mathbf{E}$.

Electric Field Intensity

- The electric field intensity, due to any charge q, can be defined as

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon_0 R^2} \quad \text{(V/m)} \quad \text{(in free space),}$$

If any point charge q' is present in an electric field $\mathbf{E}$ (due to other charges), the point charge will experience a force acting on it equal to $\mathbf{F}_e = q'\mathbf{E}$.

- Where, R is the distance between charge and the point of observation

- $\hat{R}$ is the unit vector pointing away from the charge

Two Important Properties of Electric Charge

1. Law of Conservation of Charge: The electric charge (net) can neither be created nor destroyed

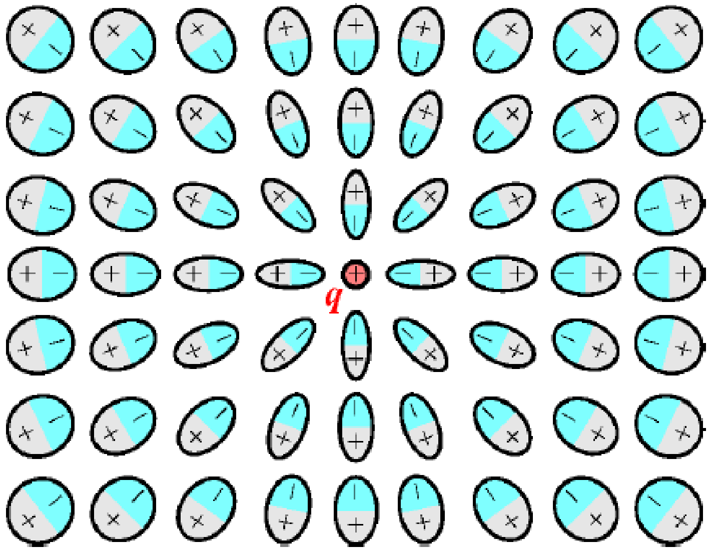
   | if a volume contains n_e number of electrons and n_p number of protons, then the total charge q is given by $q = n_p e - n_e e = (n_p - n_e)e$

2. Principle of linear superposition: the total vector electric field at a point in space due to a system of point charges is equal to the vector sum of electric fields at that point due to the individual fields.

   | Polarization of atoms of a dielectric material:

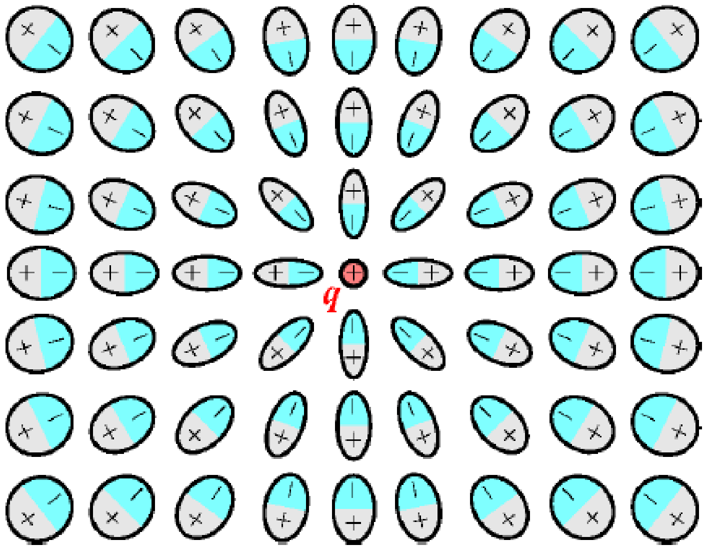
Two Important Properties of Electric Charge

- Polarization of atoms of a dielectric material due to a positive charge q: Under normal conditions, electric field at any point inside a molecule is zero. When a positive charge q is placed inside the material, the electrons of all the atoms get attracted to q, hence giving rise to electric dipoles. The overall process of electric dipole creation and their alignment is called polarization.



Electric Dipole and Polarization

- Electrons exist around the nucleus of atoms under normal conditions, but if due to any external force the electrons get concentrated on one side, leaving the rest of the charge on other side, such a polarized atom is known as electric dipole
- The intensity of polarization depends upon the distance of the atom from the test charge



Permittivity and Relative Permittivity of Material

- The electric field intensity at a point near a test charge, is different in vacuum as compared to that in a material. It is given by

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}$$

- *ε is the permittivity of the material*
- Normally the permittivity of different materials is given with reference to that of free space as $\varepsilon = \varepsilon_r \varepsilon_0 \ (F/m)$, where ε_r is the relative permittivity of that material
- For vacuum ε_r=1, while for air near earth's surface ε_r=1.0006

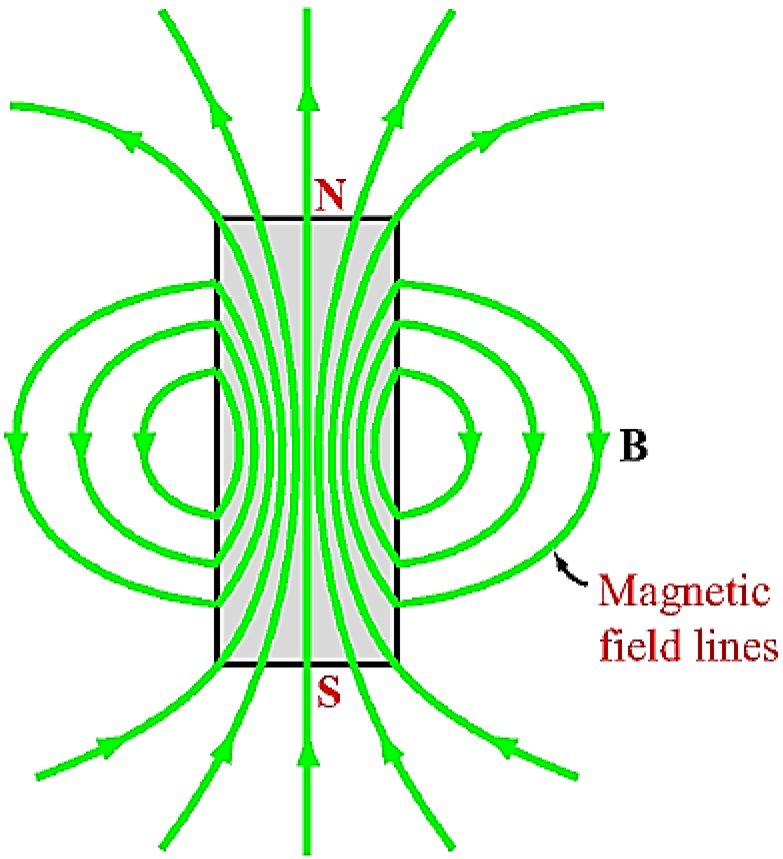
Electric Flux Density

$$D = \varepsilon E \quad (C/m^2)$$

Magnetic Field

Magnet and Magnetic Field

- Magnets occurred naturally
- The idea of magnetic field lines, their direction, and poles was presented using the compass needle as a testing instrument



Magnet and Magnetic Field

- Magnetic field lines surrounding the magnet shows the magnetic flux density represented by B
- Magnetic field is also generated by the flow of electric charge.
- The following relation, known as Biot-savart law, describes the magnitude of magnetic flux density around a current carrying conductor

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I}{2\pi r} \qquad \text{(T)}$$

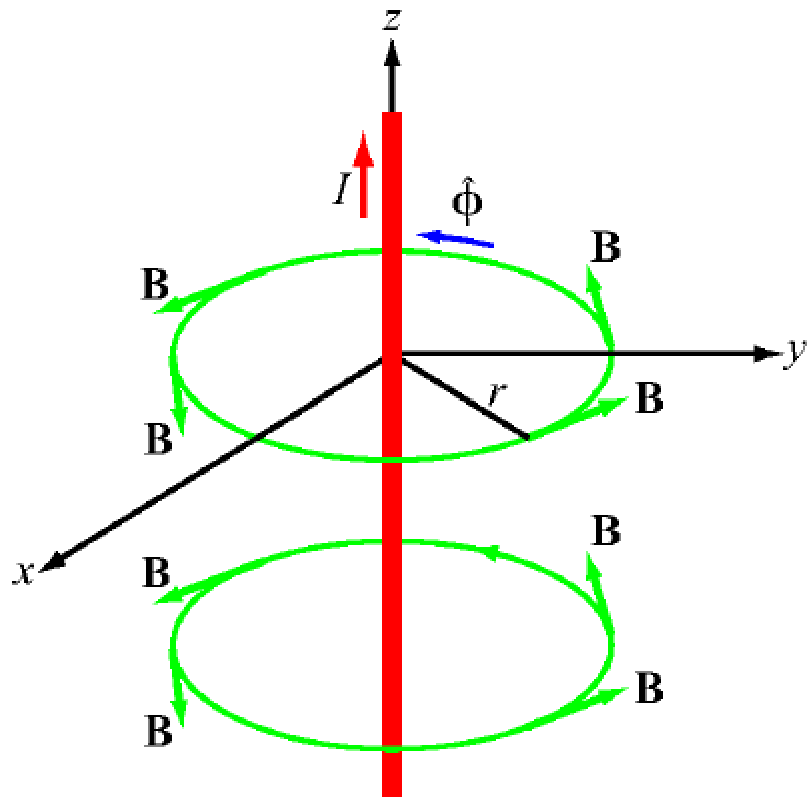
- μ_0 is called the magnetic permeability of free space and its value is

$$4\pi \text{ x } 10^{-7}\ H/m$$

It is analogous to the electric permittivity ε_0

Biot-Savart Law

- The magnetic field induced by a steady state current flowing in the z-direction



$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \qquad \text{(T)}$$

Magnetic Permeability

- As discussed for electric permittivity, for magnetic permeability we have

$$\mu = \mu_r \mu_0$$

- Also, in analogy to the relation between electric field intensity and electric flux density, we have

$$B = \mu H$$

- Where B is the magnetic flux density, while H is the magnetic field intensity

Vector Analysis

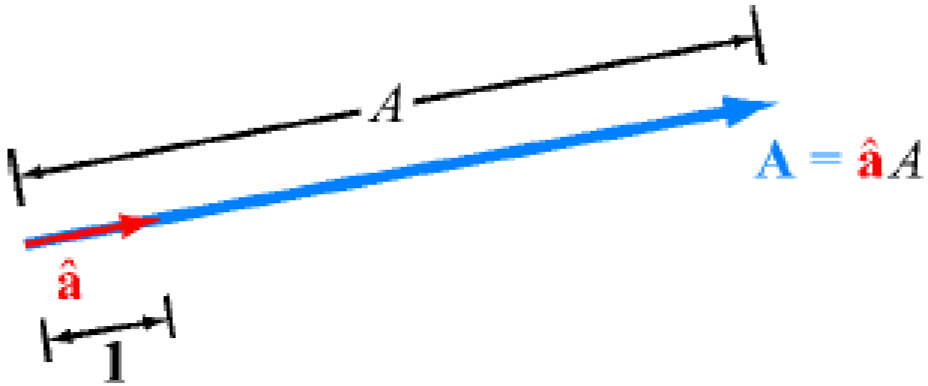
Vector Analysis

- Vector representation

- Need of vector analysis in EMF

Vector Analysis

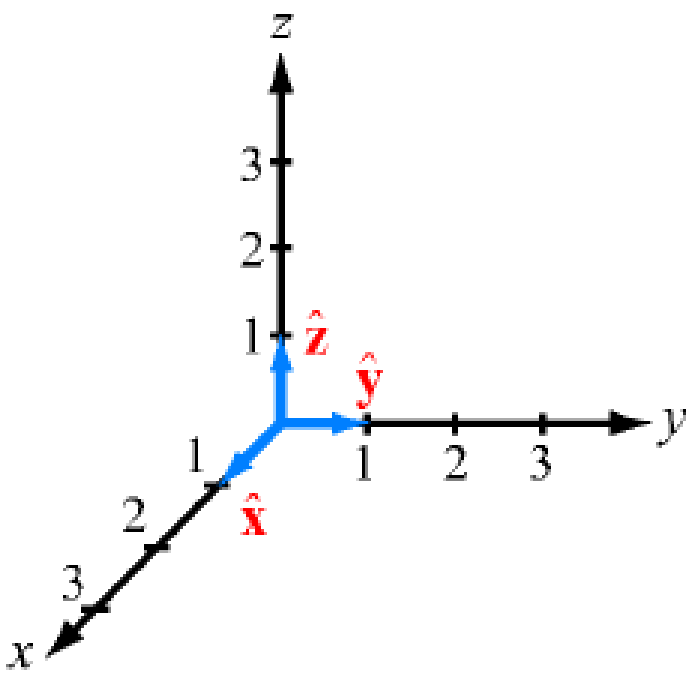
- Appearance of vector
- Magnitude and direction of vector
- Unit vector



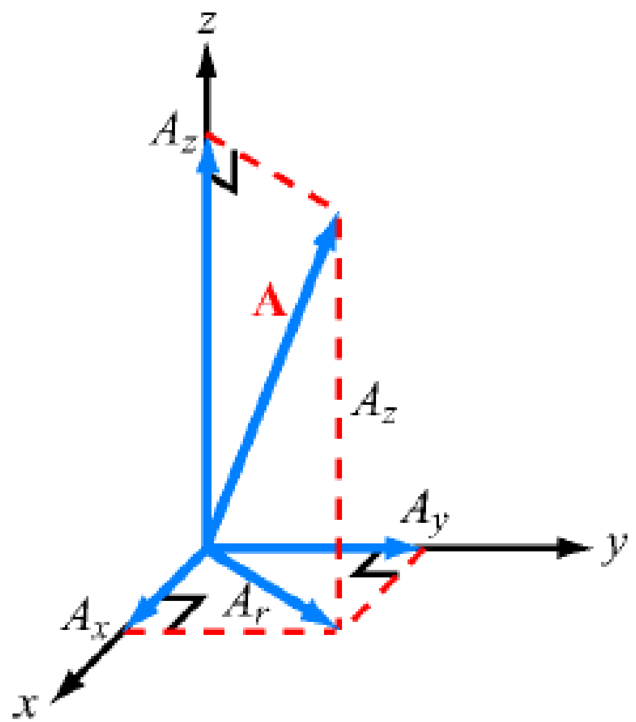
$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{a}}A$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}$$

Base Vectors and Components of Vector in Cartesian Coordinate System

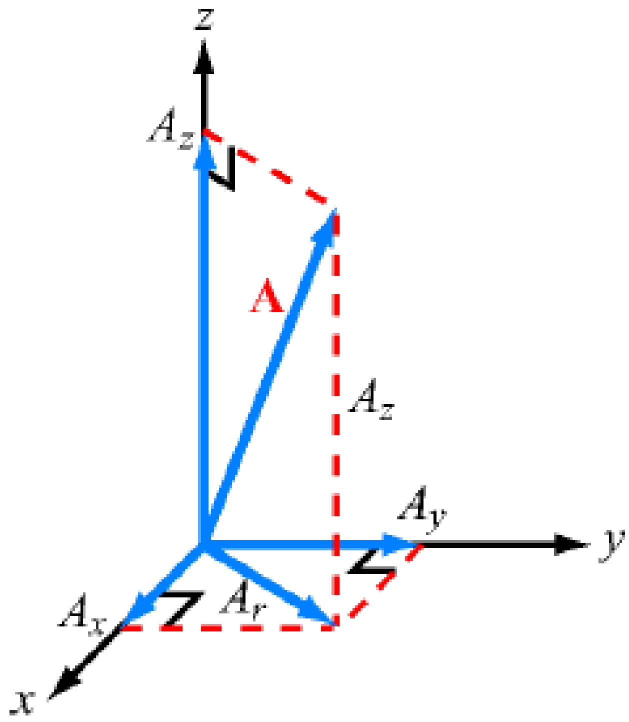


Base Vectors



Components of a Vector

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$



Components of a Vector

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

$$A = |\mathbf{A}| = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt[+]{A_x^2 + A_y^2 + A_z^2}}$$

Equality of Two Vectors

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z,$$

$$\mathbf{B} = \hat{\mathbf{b}}B = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z,$$

then $\mathbf{A} = \mathbf{B}$ if and only if $A = B$ and $\hat{\mathbf{a}} = \hat{\mathbf{b}}$, which requires that $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$.

Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another.

Addition of Two Vectors

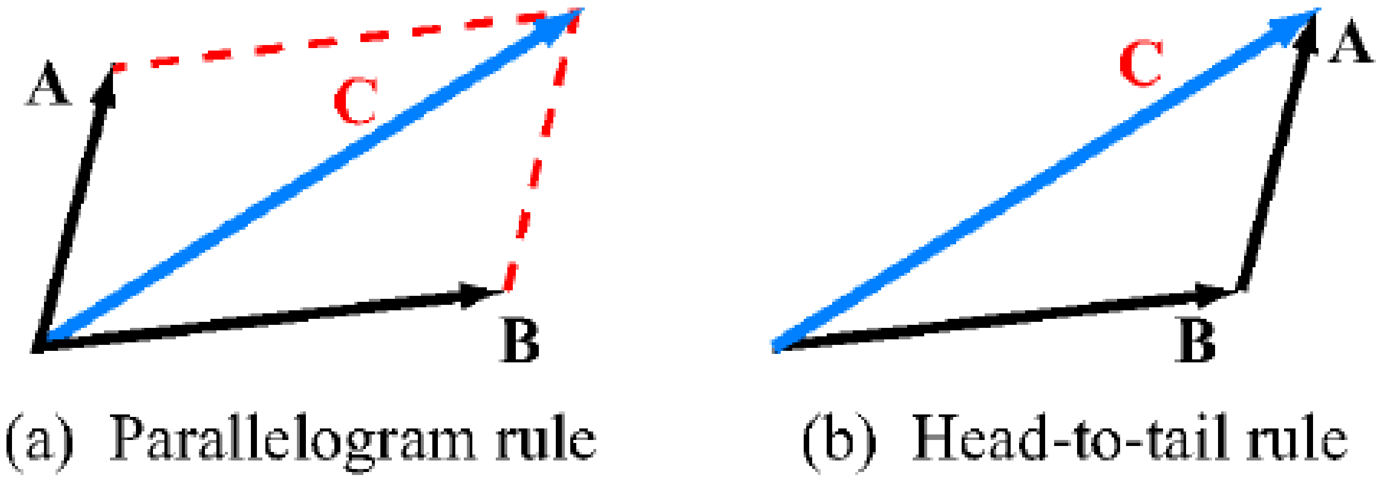
The sum of two vectors $\mathbf{A}$ and $\mathbf{B}$ is a vector $\mathbf{C} = \hat{\mathbf{x}}\,C_x + \hat{\mathbf{y}}\,C_y + \hat{\mathbf{z}}\,C_z$, given by

$$\begin{aligned}
\mathbf{C} &= \mathbf{A} + \mathbf{B} \\
&= (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) + (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \\
&= \hat{\mathbf{x}}(A_x + B_x) + \hat{\mathbf{y}}(A_y + B_y) + \hat{\mathbf{z}}(A_z + B_z) \\
&= \hat{\mathbf{x}}\,C_x + \hat{\mathbf{y}}\,C_y + \hat{\mathbf{z}}\,C_z.
\end{aligned}$$

Hence, vector addition is commutative:

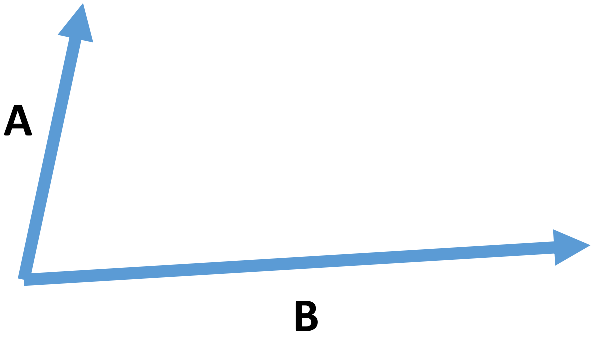
$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}.$$

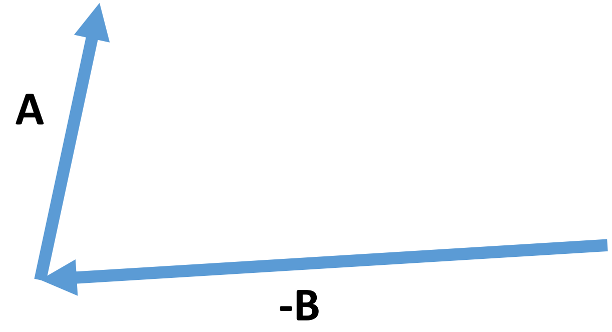
Addition of Two Vectors (Graphically)

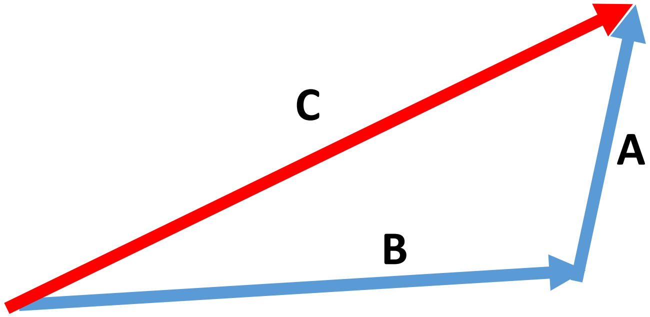


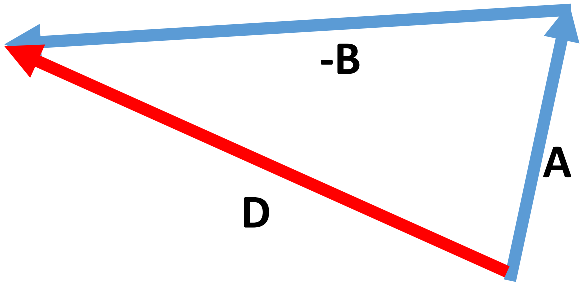
Subtraction of Vectors

$$\mathbf{D} = \mathbf{A} - \mathbf{B}$$
$$= \mathbf{A} + (-\mathbf{B})$$
$$= \hat{\mathbf{x}}(A_x - B_x) + \hat{\mathbf{y}}(A_y - B_y) + \hat{\mathbf{z}}(A_z - B_z)$$









Position and Distance Vectors

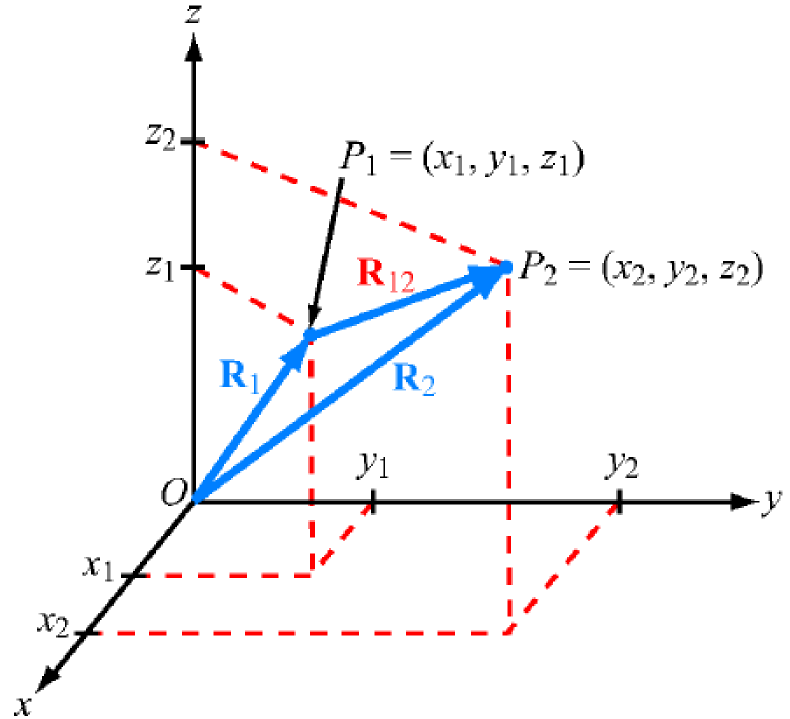
- Position vector of a point P in space is a vector from origin to point P

$$\mathbf{R}_1 = \overrightarrow{OP_1} = \hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1,$$

$$\mathbf{R}_2 = \overrightarrow{OP_2} = \hat{\mathbf{x}}x_2 + \hat{\mathbf{y}}y_2 + \hat{\mathbf{z}}z_2,$$

- Distance vector from $P1$ to $P2$ is defined as

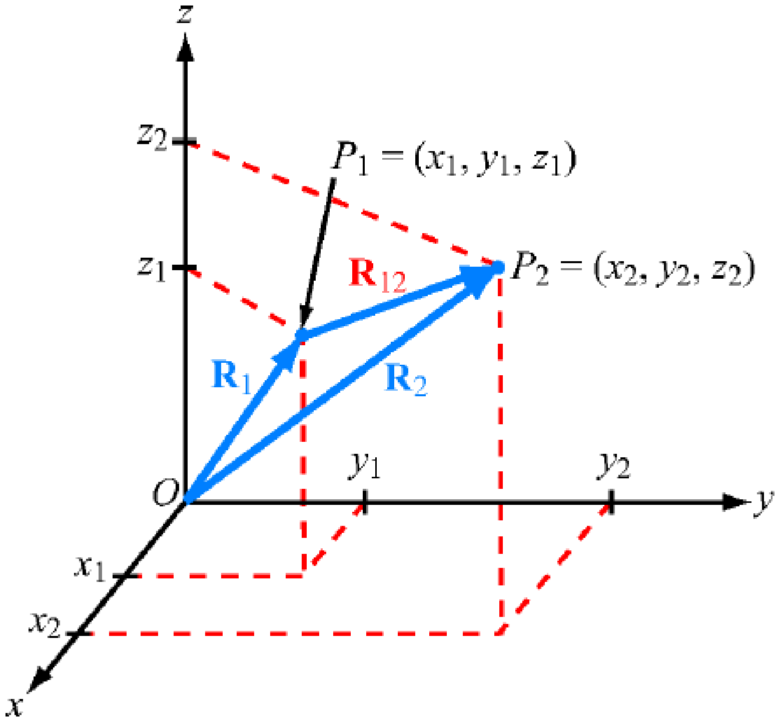
$$\begin{aligned}\mathbf{R}_{12} &= \overrightarrow{P_1P_2} \\ &= \mathbf{R}_2 - \mathbf{R}_1 \\ &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1),\end{aligned}$$



Position and Distance Vectors

- The distance d between $P1$ and $P2$ equals the magnitude of $R12$

$$d = |\mathbf{R}_{12}|$$
$$= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}.$$



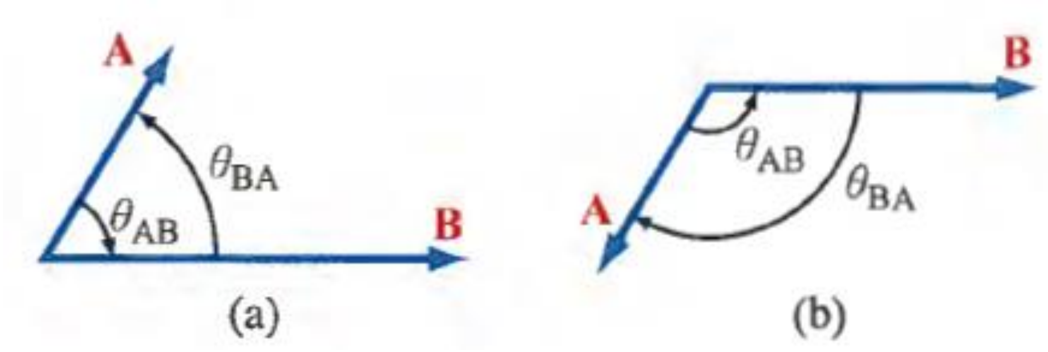
Vector Multiplication

| Simple Product: multiplying a constant k to a vector **A** to get vector **B**

$$\mathbf{B} = k\mathbf{A} = \hat{\mathbf{a}}kA = \hat{\mathbf{x}}(kA_x) + \hat{\mathbf{y}}(kA_y) + \hat{\mathbf{z}}(kA_z)$$
$$= \hat{\mathbf{x}}\,B_x + \hat{\mathbf{y}}\,B_y + \hat{\mathbf{z}}\,B_z.$$

| Scalar Product/dot product:

$$\boxed{\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}}$$



Properties of Dot Product

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(commutative property)},$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(distributive property)}.$$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2$$

$$A = |\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}}$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1,$$
$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0.$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$$\mathbf{A} \cdot \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \cdot (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z)$$

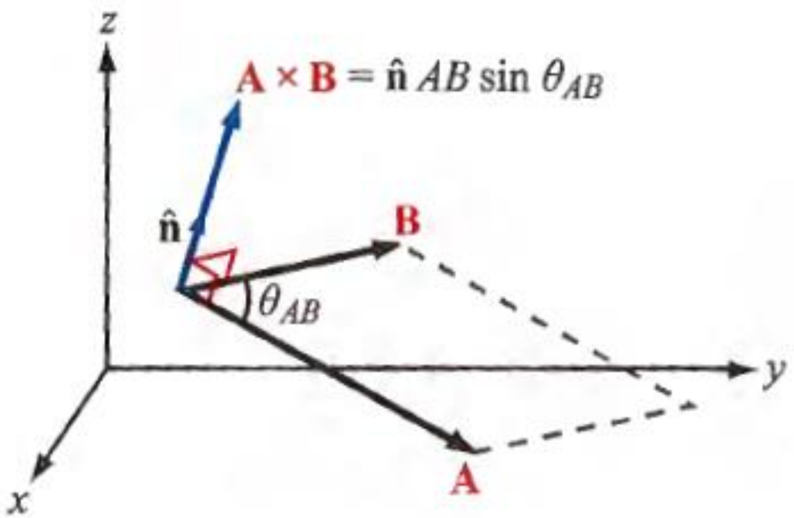
$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z.$$

Vector Multiplication

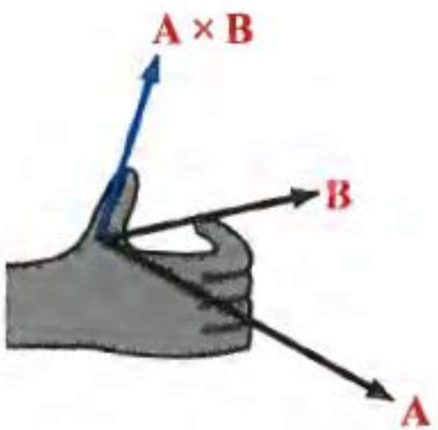
Vector/Cross Product:

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}\, AB \sin\theta_{AB}$$

where $\hat{\mathbf{n}}$ is a ***unit vector normal to the plane containing* A *and* B**



Cross product



Right-hand rule

Vector Multiplication

Properties of Vector/Cross Product:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \qquad \text{(anticommutative)}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \quad \text{(distributive)},$$

$$\mathbf{A} \times \mathbf{A} = 0.$$

$$\hat{x} \times \hat{y} = \hat{z} \qquad \hat{y} \times \hat{z} = \hat{x} \qquad \hat{z} \times \hat{x} = \hat{y}$$

$$\boxed{\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0}$$

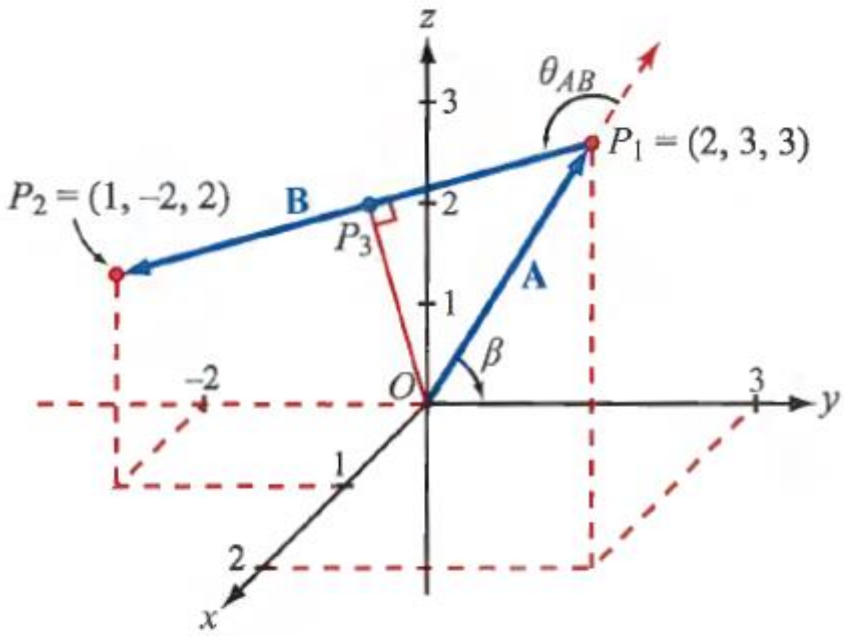
Vector Multiplication

Properties of Vector/Cross Product:

$$
\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \times (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \\
&= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) \\
&\quad + \hat{\mathbf{z}}(A_xB_y - A_yB_x).
\end{aligned}
$$

$$
\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}
$$

Example



a) Vector $\mathbf{A}$, *its magnitude, and a unit vector in the direction of* $\mathbf{A}$

$$\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3,$$

$$A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}\,,$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3)/\sqrt{22}\,.$$

Example

b) $angle\ \beta\ between\ Vector\ \mathbf{A}\ and\ y-axis$

$$\mathbf{A}\cdot\hat{\mathbf{y}} = |\mathbf{A}||\hat{\mathbf{y}}|\cos\beta = A\cos\beta,$$

or

$$\beta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\hat{\mathbf{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 50.2^\circ$$

c) $Vector\ \mathbf{B}$

$$\mathbf{B} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}.$$

d) $Angle\ \theta_{AB}\ between\ \mathbf{A}\ and\ \mathbf{B}$

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right] = \cos^{-1}\left[\frac{(-2-15-3)}{\sqrt{22}\sqrt{27}}\right]$$

$$= 145.1^\circ.$$

Example

e) *the perpendicular distance from origin to vector B*

$$|\overrightarrow{OP_3}| = |\mathbf{A}| \sin(180^\circ - \theta_{AB})$$
$$= \sqrt{22}\ \sin(180^\circ - 145.1^\circ) = 2.68.$$

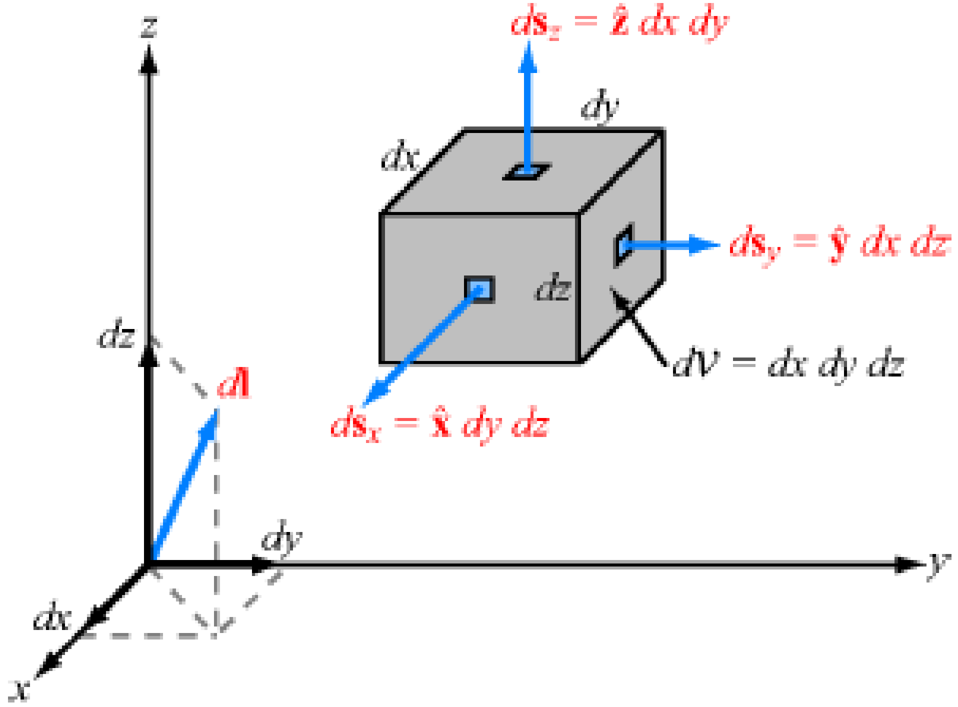
Coordinate Systems

Coordinate Systems

- Orthogonal
  - Cartesian
  - Cylindrical
  - Spherical
- Unorthogonal

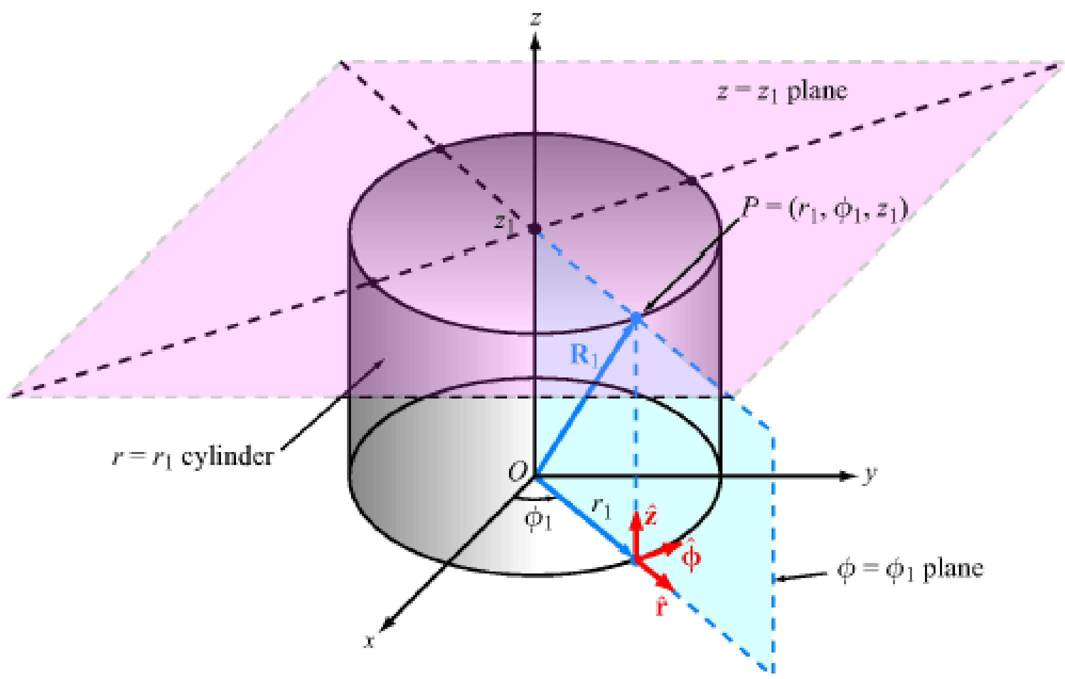
Cartesian Coordinate System

- Differential lengths
- Differential surface
- Differential volume



Cylindrical Coordinate System

- A point is defined by three variables $r, \emptyset, z$
- r is the radial distance from the z-axis
- $\emptyset$ is the azimuth angle as measured with reference to the positive x-axis
- z is the same as that in Cartesian coordinate system



Cylindrical Coordinate System

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \qquad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$$

$$\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0.$$

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{r}} A_r + \hat{\boldsymbol{\phi}} A_\phi + \hat{\mathbf{z}} A_z$$

$$|\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}} = \sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}\,.$$

Cylindrical Coordinate System

$$dl_r = dr, \qquad dl_\phi = r\, d\phi, \qquad dl_z = dz.$$

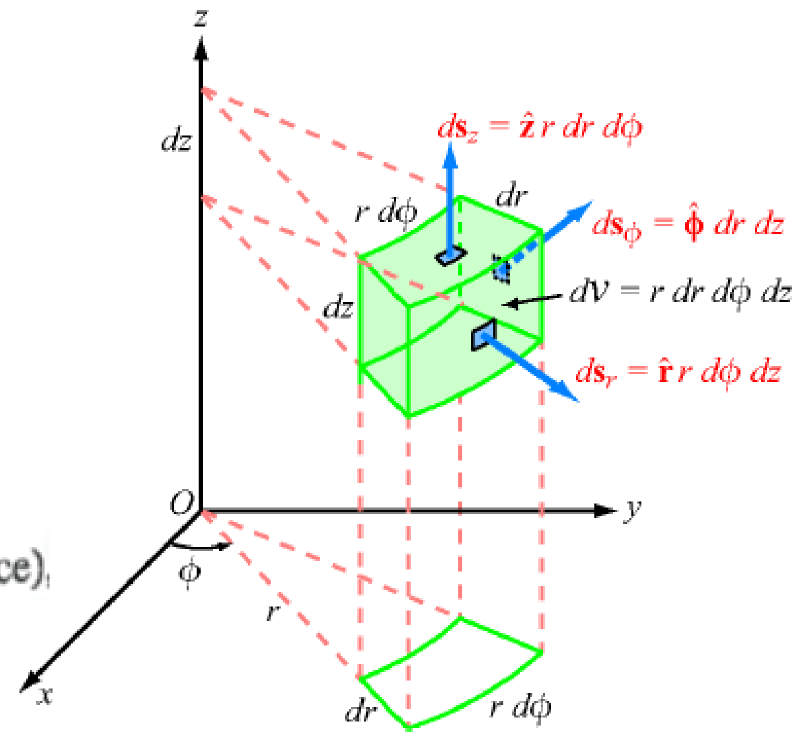
$$d\mathbf{l} = \hat{\mathbf{r}}\, dl_r + \hat{\boldsymbol{\phi}}\, dl_\phi + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{r}}\, dr + \hat{\boldsymbol{\phi}} r\, d\phi + \hat{\mathbf{z}}\, dz.$$

$$d\mathbf{s}_r = \hat{\mathbf{r}}\, dl_\phi\, dl_z = \hat{\mathbf{r}} r\, d\phi\, dz \quad (\phi\text{–}z \text{ cylindrical surface}),$$

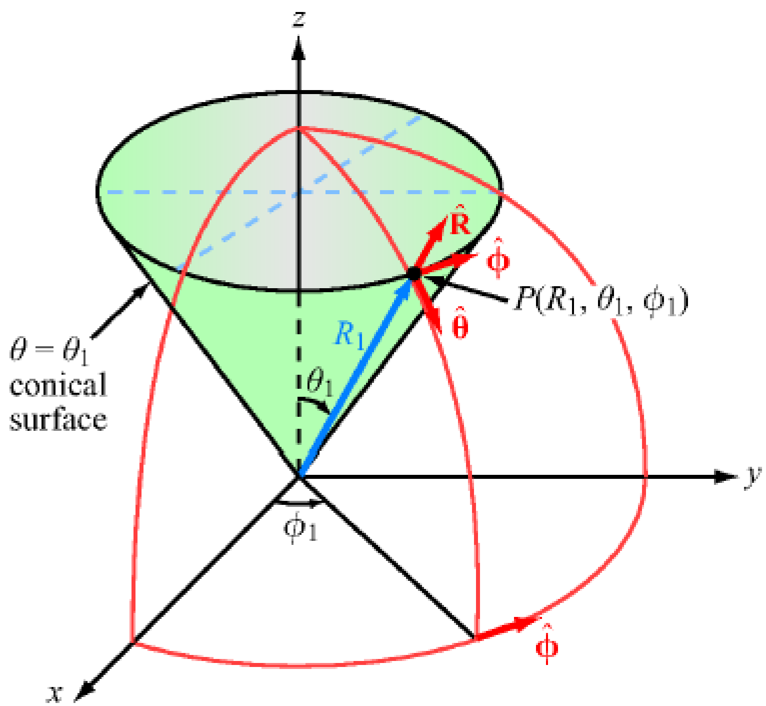
$$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, dl_r\, dl_z = \hat{\boldsymbol{\phi}}\, dr\, dz \quad (r\text{–}z \text{ plane}),$$

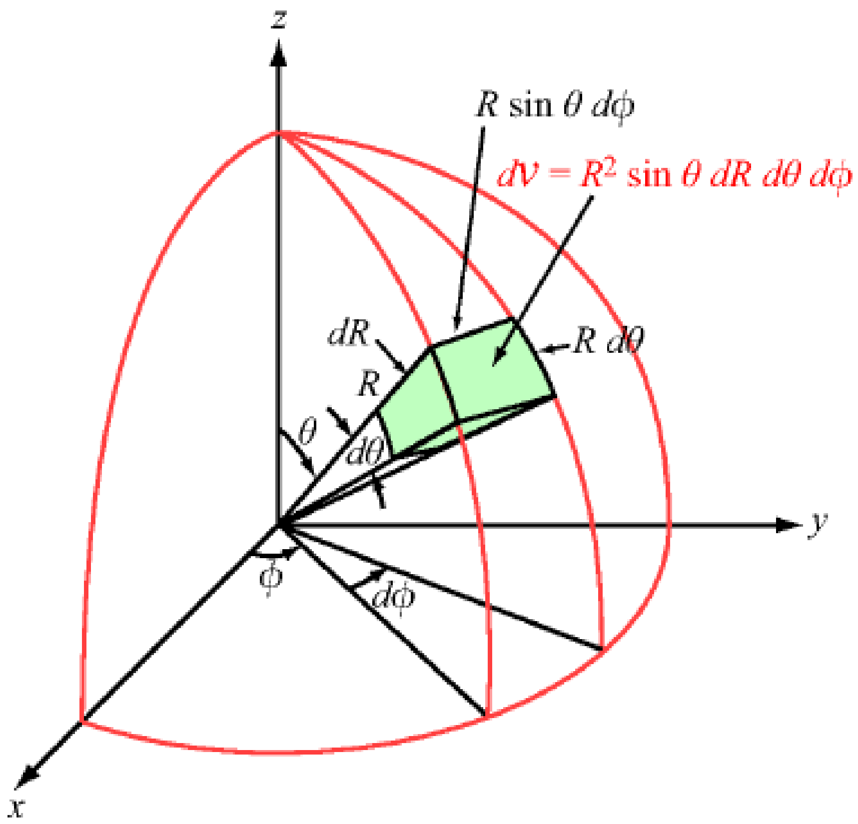
$$d\mathbf{s}_z = \hat{\mathbf{z}}\, dl_r\, dl_\phi = \hat{\mathbf{z}} r\, dr\, d\phi \quad (r\text{–}\phi \text{ plane}).$$

$$dV = dl_r\, dl_\phi\, dl_z = r\, dr\, d\phi\, dz.$$



Spherical Coordinate System





Gradient of a Scalar Field

$$\nabla T = \text{grad } T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$$

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Gradient of a Scalar Field

the gradient operator itself has no physical meaning, it attains a physical meaning once it operates on a scalar quantity, and the result of the operation is a vector with magnitude equal to the maximum rate of change of the physical quantity per unit distance and pointing in the direction of maximum increase.

Directional Derivative

With $d\mathbf{l} = \hat{\mathbf{a}}_l dl$, where $\hat{\mathbf{a}}_l$ is the unit vector of $d\mathbf{l}$, the ***directional derivative*** of T along $\hat{\mathbf{a}}_l$ is

$$\boxed{\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l}$$

We can find the difference $(T_2 - T_1)$, where $T_1 = T(x_1, y_1, z_1)$ and $T_2 = T(x_2, y_2, z_2)$ are the values of T at points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ not necessarily infinitesimally close to one another, by integrating both sides of Eq. (3.73). Thus,

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot d\mathbf{l}.$$

Directional Derivative – Example

Find the directional derivative of $T = x^2 + y^2 z$ along direction $\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2$ and evaluate it at $(1, -1, 2)$.

Solution: First, we find the gradient of T:

$$\nabla T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(x^2 + y^2 z)$$

$$= \hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2.$$

We denote $\mathbf{l}$ as the given direction,

$$\mathbf{l} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2.$$

Its unit vector is

$$\hat{\mathbf{a}}_l = \frac{\mathbf{l}}{|\mathbf{l}|} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}.$$

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l = (\hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2) \cdot \left(\frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}\right)$$

$$= \frac{4x + 6yz - 2y^2}{\sqrt{17}}.$$

Directional Derivative – Example

At (1, −1, 2),

$$\left.\frac{dT}{dl}\right|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = \frac{-10}{\sqrt{17}}$$

Properties of Gradient Operator

For any two scalar functions U and V, the following properties apply to gradient

(1) $\nabla(U+V) = \nabla U + \nabla V,$

(2) $\nabla(UV) = U\,\nabla V + V\,\nabla U,$

(3) $\nabla V^n = nV^{n-1}\,\nabla V, \qquad \text{for any } n.$

Curl of a Vector

Electrostatics

Maxwell's Equations

The modern theory of electromagnetism is based on a set of four fundamental relations known as ***Maxwell's equations***:

$$\nabla \cdot \mathbf{D} = \rho_v, \qquad (4.1a)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad (4.1b)$$

$$\nabla \cdot \mathbf{B} = 0, \qquad (4.1c)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \qquad (4.1d)$$

Here **E** and **D** are the ***electric field intensity*** and ***electric flux density***, interrelated by $\mathbf{D} = \varepsilon \mathbf{E}$ where ε is the electrical permittivity; **H** and **B** are the ***magnetic field intensity*** and ***magnetic flux density***, interrelated by $\mathbf{B} = \mu \mathbf{H}$ where μ is the magnetic permeability; ρ_v is the electric charge density per unit volume; and **J** is the current density per unit area.

Static Fields

Under ***static*** conditions, none of the quantities appearing in Maxwell's equations are functions of time (i.e., $\partial/\partial t = 0$). ***This happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that ρ_v and J are constant in time.***

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_v, \quad (4.2a)$$

$$\nabla \times \mathbf{E} = 0. \quad (4.2b)$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0, \quad (4.3a)$$

$$\nabla \times \mathbf{H} = \mathbf{J}. \quad (4.3b)$$

Static Fields

Electric and magnetic fields become decoupled under static conditions.

Volume Charge Density

$$\rho_v = \lim_{\Delta \mathcal{V} \to 0} \frac{\Delta q}{\Delta \mathcal{V}} = \frac{dq}{d\mathcal{V}} \qquad (\text{C/m}^3),$$

where Δq is the charge contained in $\Delta \mathcal{V}$. In general, ρ_v depends on spatial location (x, y, z) and t; thus, $\rho_v = \rho_v(x, y, z, t)$. Physically, ρ_v represents the average charge per unit volume for a volume $\Delta \mathcal{V}$ centered at (x, y, z), with $\Delta \mathcal{V}$ being large enough to contain a large number of atoms, yet small enough to be regarded as a point at the macroscopic scale under consideration. The variation of ρ_v with spatial location is called its ***spatial distribution***, or simply its ***distribution***. The total charge contained in volume $\mathcal{V}$ is

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} \qquad (\text{C}).$$

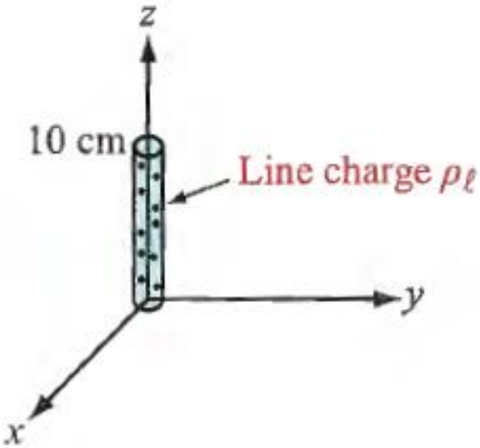
Surface and line Charge Density

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \qquad (\text{C/m}^2),$$

$$\rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \qquad (\text{C/m}).$$

Example

Calculate the total charge Q contained in a cylindrical tube oriented along the z-axis as shown in Fig. 4-1(a). The line



(a) Line charge distribution

charge density is $\rho_\ell = 2z$, where z is the distance in meters from the bottom end of the tube. The tube length is 10 cm.

Solution: The total charge Q is

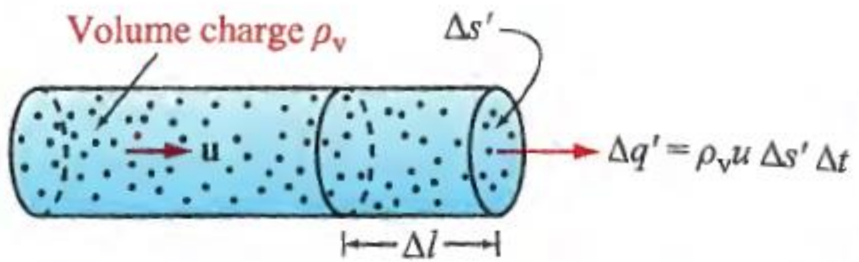
$$Q = \int_0^{0.1} \rho_\ell \, dz = \int_0^{0.1} 2z \, dz = z^2 \Big|_0^{0.1} = 10^{-2} \text{ C}.$$

Exercise 4-1: A square plate residing in the x–y plane is situated in the space defined by $-3 \text{ m} \leq x \leq 3 \text{ m}$ and $-3 \text{ m} \leq y \leq 3 \text{ m}$. Find the total charge on the plate if the surface charge density is $\rho_s = 4y^2$ (μC/m^2).

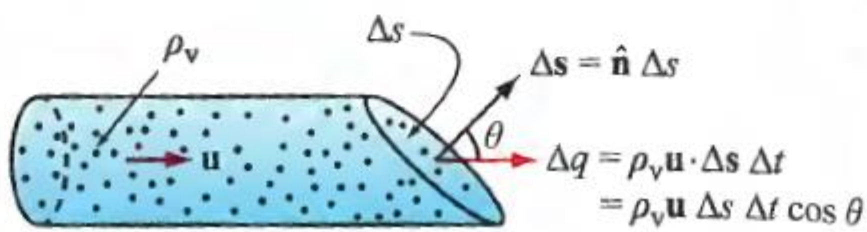
Answer: $Q = 0.432$ (mC). (See)

Consider a tube with volume charge density ρ_v
The charges in the tube move with velocity $\mathbf{u}$ along the tube axis.
Over a period Δt, the charges move a distance $\Delta l = u\ \Delta t$.
The amount of charge that crosses the tube's cross-sectional surface $\Delta s'$ in time Δt is therefore

$$\Delta q' = \rho_v\ \Delta \mathcal{V} = \rho_v\ \Delta l\ \Delta s' = \rho_v u\ \Delta s'\ \Delta t .$$



(a)



(b)

Current Density

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s},$$

where

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2) \qquad (4.11)$$

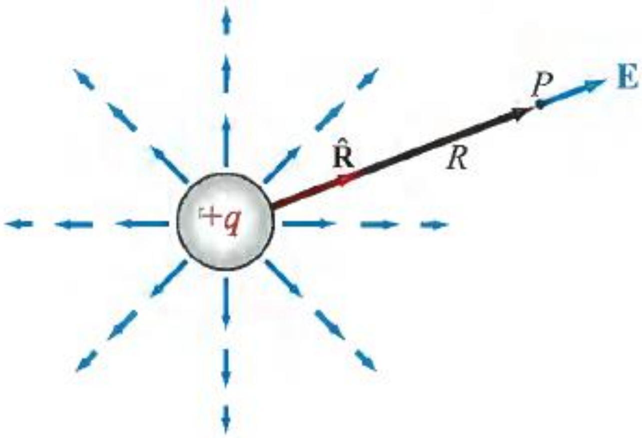
is defined as the ***current density*** in ampere per square meter. Generalizing to an arbitrary surface S, the total current flowing through it is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad (\text{A}). \qquad (4.12)$$

(1) An isolated charge q induces an electric field $\mathbf{E}$ at every point in space, and at any specific point P, $\mathbf{E}$ is given by

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}, \qquad (4.13)$$

where $\hat{\mathbf{R}}$ is a unit vector pointing from q to P (Fig. 4-3), R is the distance between them, and ε is the electrical permittivity of the medium containing the observation point P.



(2) In the presence of an electric field $\mathbf{E}$ at a given point in space, which may be due to a single charge or a distribution of charges, the force acting on a test charge q' when placed at P, is

$$\mathbf{F} = q'\mathbf{E} \qquad \text{(N)}. \qquad (4.14)$$

With $\mathbf{F}$ measured in newtons (N) and q' in coulombs (C), the unit of $\mathbf{E}$ is (N/C), which will be shown later in Section 4-5 to be the same as volt per meter (V/m).

Electric Field Due to Multiple Point Charges

with R, the distance between q_1 and P, replaced with $|\mathbf{R} - \mathbf{R}_1|$ and the unit vector $\hat{\mathbf{R}}$ replaced with $(\mathbf{R} - \mathbf{R}_1)/|\mathbf{R} - \mathbf{R}_1|$. Thus

$$\mathbf{E}_1 = \frac{q_1(\mathbf{R} - \mathbf{R}_1)}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_1|^3} \quad \text{(V/m)}. \tag{4.17a}$$

Similarly, the electric field at P due to q_2 alone is

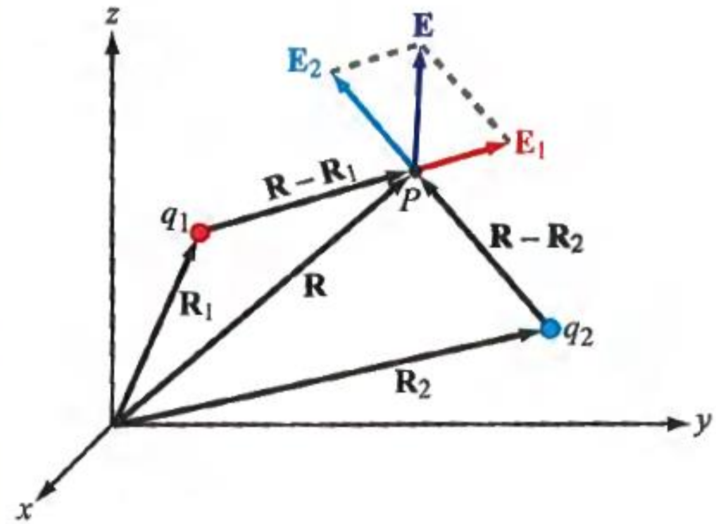
$$\mathbf{E}_2 = \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_2|^3} \quad \text{(V/m)}. \tag{4.17b}$$

The electric field obeys the principle of linear superposition.

Hence, the total electric field $\mathbf{E}$ at P due to q_1 and q_2 is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

$$= \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R} - \mathbf{R}_1)}{|\mathbf{R} - \mathbf{R}_1|^3} + \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{|\mathbf{R} - \mathbf{R}_2|^3}\right]. \tag{4.18}$$



Electric Field Due to Multiple Point Charges

Generalizing the preceding result to the case of N point charges, the electric field $\mathbf{E}$ at point P with position vector $\mathbf{R}$ due to charges $q_1, q_2, \ldots, q_N$ located at points with position vectors $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N$, equals the vector sum of the electric fields induced by all the individual charges, or

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \quad \text{(V/m)}. \qquad (4.19)$$

Example 4-3: Electric Field Due to Two Point Charges

Two point charges with $q_1 = 2 \times 10^{-5}$ C and $q_2 = -4 \times 10^{-5}$ C are located in free space at points with Cartesian coordinates $(1, 3, -1)$ and $(-3, 1, -2)$, respectively. Find (a) the electric field **E** at $(3, 1, -2)$ and (b) the force on a 8×10^{-5} C charge located at that point. All distances are in meters.

Solution: **(a)** From Eq. (4.18), the electric field **E** with $\varepsilon = \varepsilon_0$ (free space) is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[q_1 \frac{(\mathbf{R} - \mathbf{R}_1)}{|\mathbf{R} - \mathbf{R}_1|^3} + q_2 \frac{(\mathbf{R} - \mathbf{R}_2)}{|\mathbf{R} - \mathbf{R}_2|^3}\right] \quad \text{(V/m)}$$

The vectors $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}$ are

$$\mathbf{R}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}},$$

$$\mathbf{R}_2 = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2,$$

$$\mathbf{R} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2.$$

Electric Field Due to Multiple Point Charges

Hence,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2(\hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}})}{27} - \frac{4(\hat{\mathbf{x}}6)}{216}\right] \times 10^{-5}$$

$$= \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{108\pi\varepsilon_0} \times 10^{-5} \qquad \text{(V/m)}.$$

(b)

$$\mathbf{F} = q_3\mathbf{E} = 8 \times 10^{-5} \times \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{108\pi\varepsilon_0} \times 10^{-5}$$

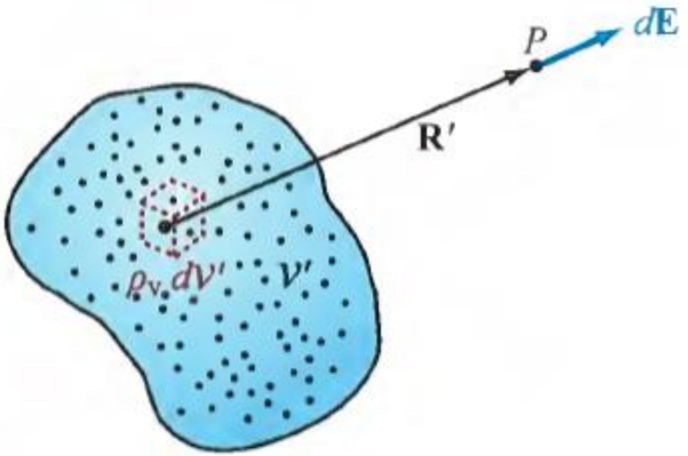
$$= \frac{\hat{\mathbf{x}}2 - \hat{\mathbf{y}}8 - \hat{\mathbf{z}}4}{27\pi\varepsilon_0} \times 10^{-10} \qquad \text{(N)}.$$

Electric Field Due to a Charge Distribution

We now extend the results obtained for the field due to discrete point charges to continuous charge distributions. Consider a volume $\mathcal{V}'$ that contains a distribution of electric charge with volume charge density ρ_v, which may vary spatially within $\mathcal{V}'$ (Fig. 4-5). The differential electric field at a point P due to

a differential amount of charge $dq = \rho_v \, d\mathcal{V}'$ contained in a differential volume $d\mathcal{V}'$ is

$$d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi\varepsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{4\pi\varepsilon R'^2},$$



where $\mathbf{R}'$ is the vector from the differential volume $d\mathcal{V}'$ to point P. Applying the principle of linear superposition, the total electric field $\mathbf{E}$ is obtained by integrating the fields due to all differential charges in $\mathcal{V}'$. Thus,

$$\mathbf{E} = \int_{\mathcal{V}'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{R'^2}$$

(volume distribution). (4.21a)

It is important to note that, in general, both R' and $\hat{\mathbf{R}}'$ vary as a function of position over the integration volume $\mathcal{V}'$.

Electric Field Due to a Charge Distribution

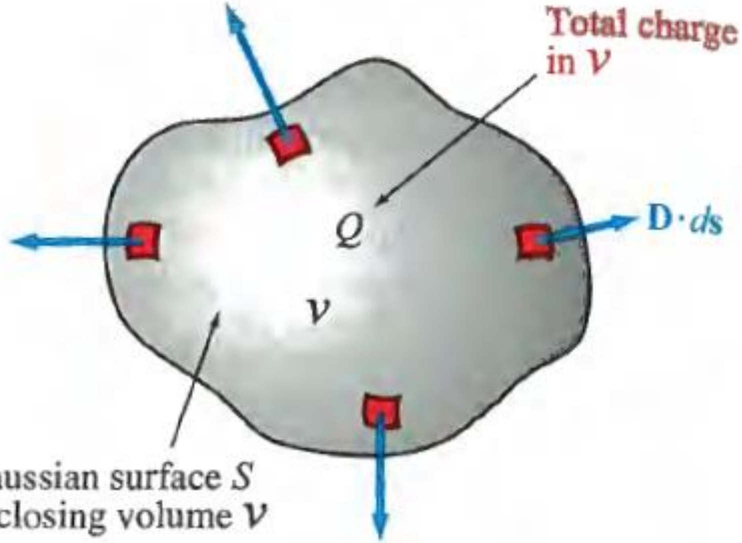
If the charge is distributed across a surface S' with surface charge density ρ_s, then $dq = \rho_s \, ds'$, and if it is distributed along a line l' with a line charge density ρ_ℓ, then $dq = \rho_\ell \, dl'$. Accordingly, the electric fields due to surface and line charge distributions are

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2} \quad \text{(surface distribution)}, \tag{4.21b}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2} \quad \text{(line distribution)}. \tag{4.21c}$$

Gauss's Law

The integral form of Gauss's law is illustrated diagrammatically in Fig. 4-8; for each differential surface element ds, **D** $\cdot$ ***ds*** *is the electric field flux flowing outward of V through ds, and the total flux through surface S equals the enclosed charge Q. The surface S is called a* ***Gaussian surface.***



$$\oint_S \mathbf{D}\cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).

Electric Scalar Potential

The term "voltage" is short for "voltage potential" and synonymous with ***electric potential.***

Electric Potential as a Function of Electric Field

We begin by considering the simple case of a positive charge q in a uniform electric field $\mathbf{E} = -\hat{\mathbf{y}}E$, in the $-y$-direction (Fig. 4-11). The presence of the field $\mathbf{E}$ exerts a force $\mathbf{F}_e = q\mathbf{E}$

on the charge in the $-y$-direction. To move the charge along the $+y$-direction (against the force $\mathbf{F}_e$), we need to provide an external force $\mathbf{F}_{ext}$ to counteract $\mathbf{F}_e$, which requires the expenditure of energy. To move q without acceleration (at constant speed), the net force acting on the charge must be zero, which means that $\mathbf{F}_{ext} + \mathbf{F}_e = 0$, or

$$\mathbf{F}_{ext} = -\mathbf{F}_e = -q\mathbf{E}. \tag{4.34}$$

The work done, or energy expended, in moving any object a vector differential distance $d\mathbf{l}$ while exerting a force $\mathbf{F}_{ext}$ is

$$dW = \mathbf{F}_{ext} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J)}. \tag{4.35}$$

Work, or energy, is measured in joules (J). If the charge is moved a distance dy along $\hat{\mathbf{y}}$, then

$$dW = -q(-\hat{\mathbf{y}}E) \cdot \hat{\mathbf{y}}\, dy = qE\, dy. \tag{4.36}$$

Electric Potential as a Function of Electric Field

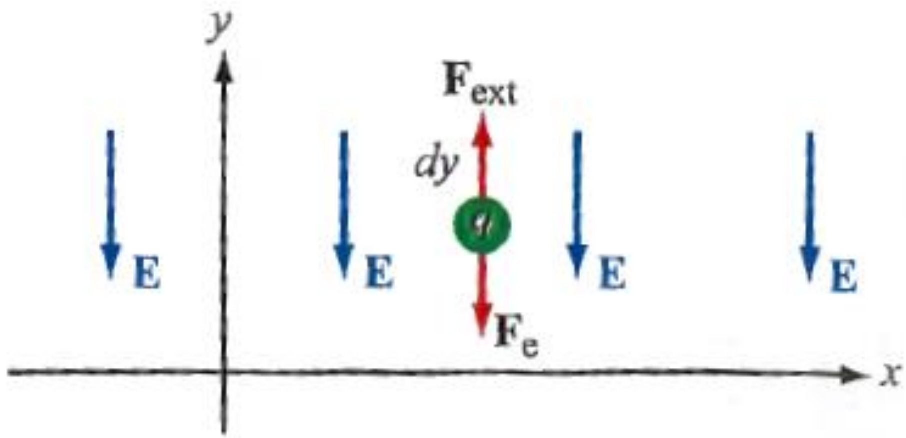


Figure 4-11: Work done in moving a charge q a distance dy against the electric field **E** is $dW = qE\,dy$.

Electric Potential as a Function of Electric Field

The differential electric potential energy dW per unit charge is called the ***differential electric potential*** (or differential voltage) dV. That is,

$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J/C or V).} \qquad (4.37)$$

The unit of V is the volt (V), with 1 V = 1 J/C, and since V is measured in volts, the electric field is expressed in volts per meter (V/m).

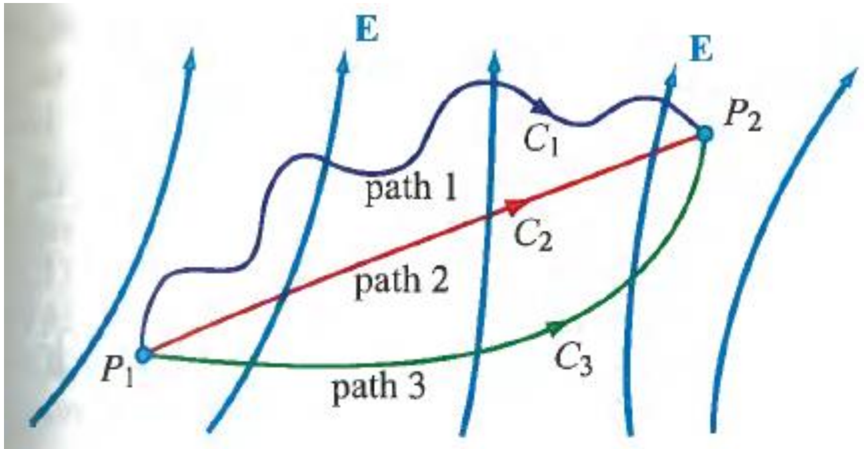


Figure 4-12: In electrostatics, the potential difference between P_2 and P_1 is the same irrespective of the path used for calculating the line integral of the electric field between them.

Electric Potential as a Function of Electric Field

The ***potential difference*** corresponding to moving a point charge from point P_1 to point P_2 (Fig. 4-12) is obtained by integrating Eq. (4.37) ***along any path*** between them. That is,

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \tag{4.38}$$

or

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \tag{4.39}$$

Electric Potential Due to Point Charges

The electric field due to a point charge q located at the origin is given by Eq. (4.32) as

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}. \tag{4.44}$$

The field is radially directed and decays quadratically with the distance R from the observer to the charge.

As was stated earlier, the choice of integration path between the end points in Eq. (4.43) is arbitrary. Hence, we can conveniently choose the path to be along the radial direction $\hat{\mathbf{R}}$, in which case $d\mathbf{l} = \hat{\mathbf{R}}\, dR$ and

$$V = -\int_{\infty}^{R} \left(\hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \right) \cdot \hat{\mathbf{R}}\, dR = \frac{q}{4\pi\varepsilon R} \qquad \text{(V)}. \tag{4.45}$$

If the charge q is at a location other than the origin, say at position vector $\mathbf{R}_1$, then V at observation position vector $\mathbf{R}$ becomes

$$V = \frac{q}{4\pi\varepsilon |\mathbf{R} - \mathbf{R}_1|} \qquad \text{(V)}, \tag{4.46}$$

where $|\mathbf{R} - \mathbf{R}_1|$ is the distance between the observation point and the location of the charge q. The principle of superposition applied previously to the electric field $\mathbf{E}$ also applies to the electric potential V. Hence, for N discrete point charges $q_1, q_2, \ldots, q_N$ residing at position vectors $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N$, the electric potential is

$$V = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{R} - \mathbf{R}_i|} \quad \text{(V).} \qquad (4.47)$$

Electric Potential Due to Continuous Distributions

To obtain expressions for the electric potential V due to continuous charge distributions over a volume $\mathcal{V}'$, across a surface S', or along a line l', we (1) replace q_i in Eq. (4.47) with $\rho_v\, d\mathcal{V}'$, $\rho_s\, ds'$, and $\rho_\ell\, dl'$, respectively; (2) convert the summation into an integration; and (3) define $R' = |\mathbf{R} - \mathbf{R}_i|$ as the distance between the integration point and the observation point. These steps lead to the following expressions:

$$V = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v}{R'}\, d\mathcal{V}' \quad \text{(volume distribution)}, \tag{4.48a}$$

$$V = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_s}{R'}\, ds' \quad \text{(surface distribution)}, \tag{4.48b}$$

$$V = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_\ell}{R'}\, dl' \quad \text{(line distribution)}. \tag{4.48c}$$

Electric Field as a Function of Electric Potential

In Section 4-5.1, we expressed V in terms of a line integral over $\mathbf{E}$. Now we explore the inverse relationship by re-examining Eq. (4.37):

$$dV = -\mathbf{E} \cdot d\mathbf{l}. \tag{4.49}$$

For a scalar function V, Eq. (3.73) gives

$$dV = \nabla V \cdot d\mathbf{l}, \tag{4.50}$$

where ∇V is the gradient of V. Comparison of Eq. (4.49) with Eq. (4.50) leads to

$$\mathbf{E} = -\nabla V. \tag{4.51}$$

This differential relationship between V and $\mathbf{E}$ allows us to determine $\mathbf{E}$ for any charge distribution by first calculating V and then taking the negative gradient of V to find $\mathbf{E}$.

An ***electric dipole*** consists of two point charges of equal magnitude but opposite polarity, separated by a distance d [Fig. 4-13(a)]. Determine V and $\mathbf{E}$ at any point P, given that P is at a distance $R \gg d$ from the dipole center, and the dipole resides in free space.

Solution: To simplify the derivation, we align the dipole along the z-axis and center it at the origin [Fig. 4-13(a)]. For the two charges shown in Fig. 4-13(a), application of Eq. (4.47) gives

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1} + \frac{-q}{R_2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right).$$

Since $d \ll R$, the lines labeled R_1 and R_2 in Fig. 4-13(a) are approximately parallel to each other, in which case the following approximations apply:

$$R_2 - R_1 \simeq d\cos\theta, \qquad R_1 R_2 \simeq R^2.$$

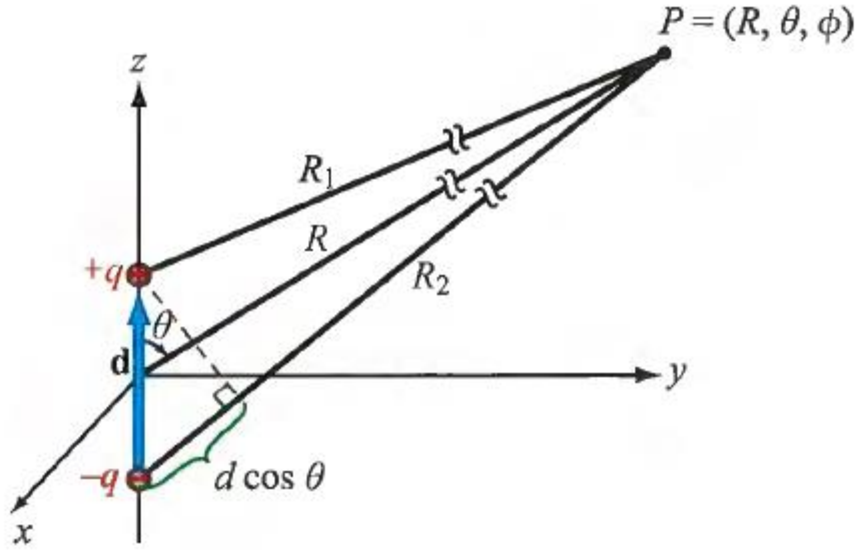
Hence,

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2}. \qquad (4.52)$$

Electric Field of an Electric Dipole

To generalize this result to an arbitrarily oriented dipole, note that the numerator of Eq. (4.52) can be expressed as the dot product of $q\mathbf{d}$ (where $\mathbf{d}$ is the distance vector from $-q$ to $+q$) and the unit vector $\hat{\mathbf{R}}$ pointing from the center of the dipole toward the observation point P. That is,

$$qd\cos\theta = q\mathbf{d}\cdot\hat{\mathbf{R}} = \mathbf{p}\cdot\hat{\mathbf{R}}, \qquad (4.53)$$



(a) Electric dipole

Electric Field of an Electric Dipole

where $\mathbf{p} = q\mathbf{d}$ is called the ***dipole moment***. Using Eq. (4.53) in Eq. (4.52) then gives

$$V = \frac{\mathbf{p} \cdot \hat{\mathbf{R}}}{4\pi\varepsilon_0 R^2} \qquad \text{(electric dipole).} \qquad (4.54)$$

Conductors

The electromagnetic ***constitutive parameters*** of a material medium are its electrical permittivity ε, magnetic permeability μ, and conductivity σ. A material is said to be ***homogeneous*** if its constitutive parameters do not vary from point to point, and ***isotropic*** if they are independent of direction. Most materials are isotropic, but some crystals are not.

The conductivity of a material is a measure of how easily electrons can travel through the material under the influence of an externally applied electric field.

Materials are classified as ***conductors*** (metals) or ***dielectrics*** (insulators) according to the magnitudes of their conductivities. A conductor has a large number of loosely attached electrons in the outermost shells of its atoms. In the absence of an external electric field, these free electrons move in random directions and with varying speeds. Their random motion produces zero average current through the conductor. Upon applying an external electric field, however, the electrons migrate from one atom to the next in the direction opposite that of the external field. Their movement gives rise to a ***conduction current***

$$\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2) \quad (\text{Ohm's law}), \qquad (4.63)$$

where σ is the material's conductivity with units of siemen per meter (S/m).

In yet other materials, called dielectrics, the electrons are tightly bound to the atoms, so much so that it is very difficult to detach them under the influence of an electric field. Consequently, no significant conduction current can flow through them.

*A **perfect dielectric** is a material with $\sigma = 0$. In contrast, a **perfect conductor** is a material with $\sigma = \infty$. Some materials, called superconductors, exhibit such a behavior.*

Dielectrics

Poisson's and Laplace's Equations

A useful approach to the calculation of electric potentials is to relate that potential to the charge density which gives rise to it

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \text{(Poisson's equation)}$$

If the medium under consideration contains no charges, Eq. reduces to

$$\nabla^2 V = 0 \qquad \text{(Laplace's equation)}$$

Magnetostatics

This chapter on magnetostatics parallels the preceding one on electrostatics. Stationary charges produce static electric fields, and steady (i.e., non-time varying) currents produce static magnetic fields. When $\partial/\partial t = 0$, the magnetic fields in a medium with magnetic permeability μ are governed by the second pair of Maxwell's equations [Eqs. (4.3a and b)]:

$$\nabla \cdot \mathbf{B} = 0, \qquad (5.1a)$$

$$\nabla \times \mathbf{H} = \mathbf{J}, \qquad (5.1b)$$

where **J** is the current density. The ***magnetic flux density*** **B** and the ***magnetic field intensity*** **H** are related by

$$\mathbf{B} = \mu \mathbf{H}. \qquad (5.2)$$

Magnetostatics

The parallelism that exists between these magnetostatic quantities and their electrostatic counterparts is elucidated in Table

Table 5-1: Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations		
• Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

Magnetic Forces and Torques

We now define the ***magnetic flux density*** **B** at a point in space in terms of the ***magnetic force*** $\mathbf{F}_m$ that acts on a charged test particle moving with velocity **u** through that point. Experiments revealed that a particle of charge q moving with velocity **u** in a magnetic field experiences a magnetic force $\mathbf{F}_m$ given by

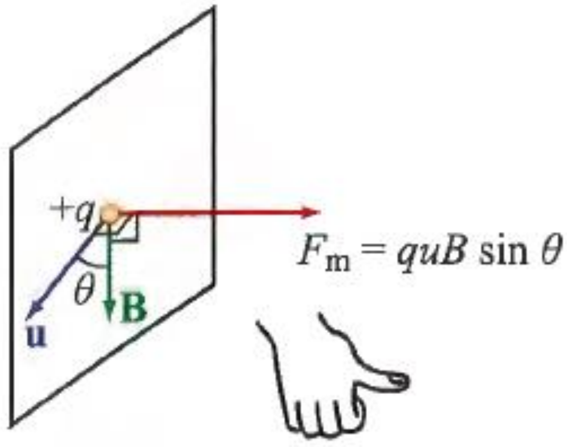
$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}. \tag{5.3}$$

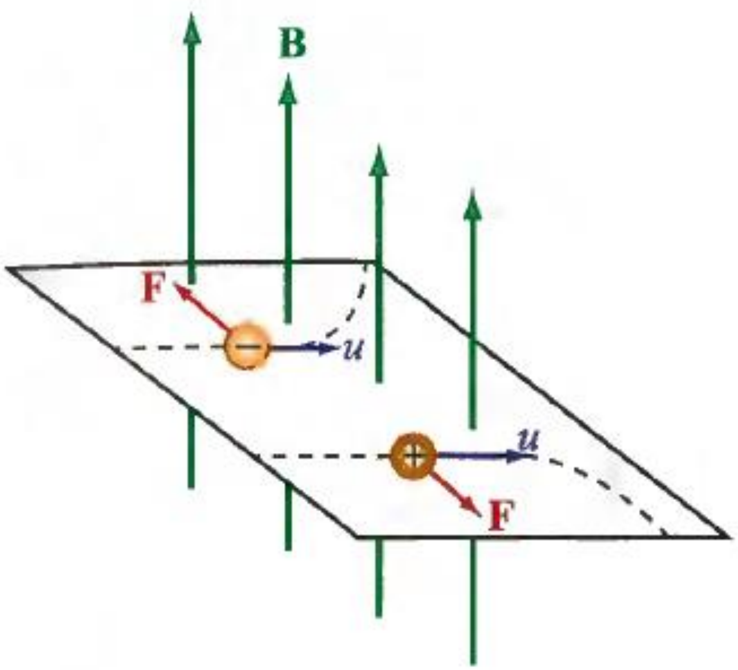
Accordingly, the strength of **B** is measured in newtons/(C·m/s), also called the tesla (T). For a positively charged particle, the direction of $\mathbf{F}_m$ is that of the cross product $\mathbf{u} \times \mathbf{B}$, which is perpendicular to the plane containing **u** and **B** and governed by the right-hand rule. If q is negative, the direction of $\mathbf{F}_m$ is reversed (Fig. 5-1). The magnitude of $\mathbf{F}_m$ is given by

$$F_m = quB \sin\theta, \tag{5.4}$$

where θ is the angle between **u** and **B**. We note that F_m is maximum when **u** is perpendicular to **B** ($\theta = 90°$), and zero when **u** is parallel to **B** ($\theta = 0$ or $180°$).

Magnetic Forces and Torques





Magnetic Forces and Torques

If a charged particle resides in the presence of both an electric field **E** and a magnetic field **B**, then the total ***electromagnetic force*** acting on it is

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \quad (5.5)$$

The force expressed by Eq. (5.5) is known as the ***Lorentz force***.

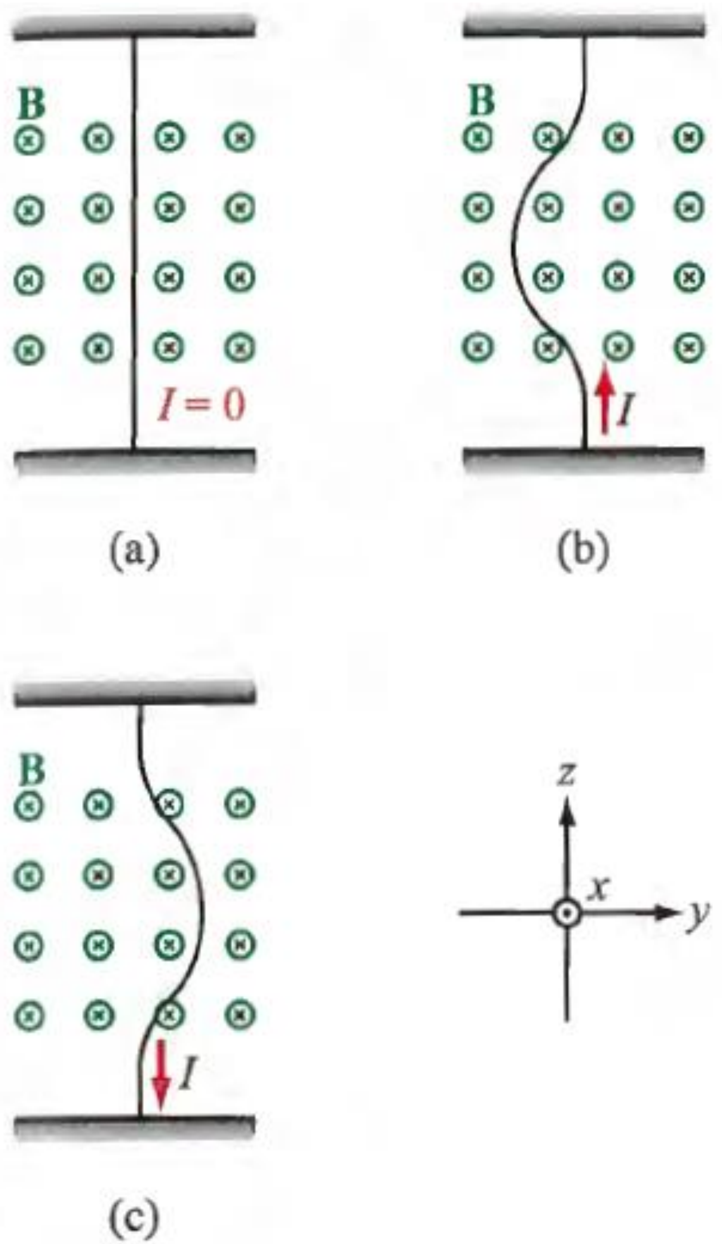
Magnetic Force on a Current Carrying Conductor

A current flowing through a conducting wire consists of charged particles *drifting* through the material of the wire. Consequently, when a current-carrying wire is placed in a magnetic field, it will experience a force equal to the sum of the magnetic forces acting on the charged particles moving within it. Consider, for example, the arrangement shown in Fig. 5-2 in which a vertical wire oriented along the z-direction is placed in a magnetic field $\mathbf{B}$ (produced by a magnet) oriented along the $-\hat{\mathbf{x}}$-direction (into the page). With no current flowing in

the wire, $\mathbf{F}_m = 0$ and the wire maintains its vertical orientation [Fig. 5-2(a)], but when a current is introduced in the wire, the wire deflects to the left ($-\hat{\mathbf{y}}$-direction) if the current direction is upward ($+\hat{\mathbf{z}}$-direction), and to the right ($+\hat{\mathbf{y}}$-direction) if the current direction is downward ($-\hat{\mathbf{z}}$-direction). The directions of these deflections are in accordance with the cross product given by Eq. (5.3).

$$\mathbf{F}_m = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}. \qquad (5.10)$$

Magnetic Force on a Current Carrying Conductor



When a force is applied on a rigid body that can pivot about a fixed axis, the body will, in general, react by rotating about that axis. The angular acceleration depends on the cross product of the applied force vector **F** and the distance vector **d**, measured *from* a point on the rotation axis (such that **d** is perpendicular to the axis) *to* the point of application of **F** (Fig. 5-5). The length of **d** is called the ***moment arm***, and the cross product

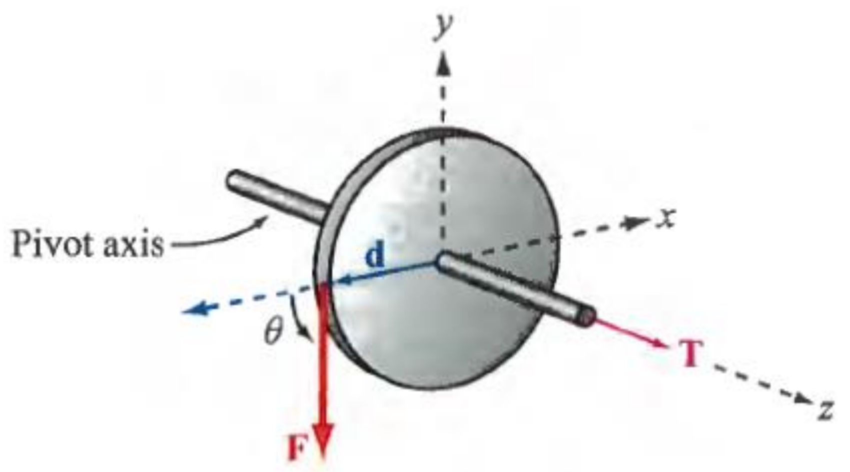
$$\mathbf{T} = \mathbf{d} \times \mathbf{F} \qquad (\text{N}\cdot\text{m}) \qquad (5.13)$$

is called the ***torque***. The unit for **T** is the same as that for work or energy, even though torque does not represent either. The force **F** applied on the disk shown in Fig. 5-5 lies in the x–y plane and makes an angle θ with **d**. Hence,

$$\mathbf{T} = \hat{\mathbf{z}} r F \sin\theta, \qquad (5.14)$$

where $|\mathbf{d}| = r$, the radius of the disk, and $F = |\mathbf{F}|$. From Eq. (5.14) we observe that a torque along the positive

Magnetic Torque on a Current Carrying Loop



z-direction corresponds to a tendency for the cylinder to rotate counterclockwise and, conversely, a torque along the $-z$-direction corresponds to clockwise rotation.

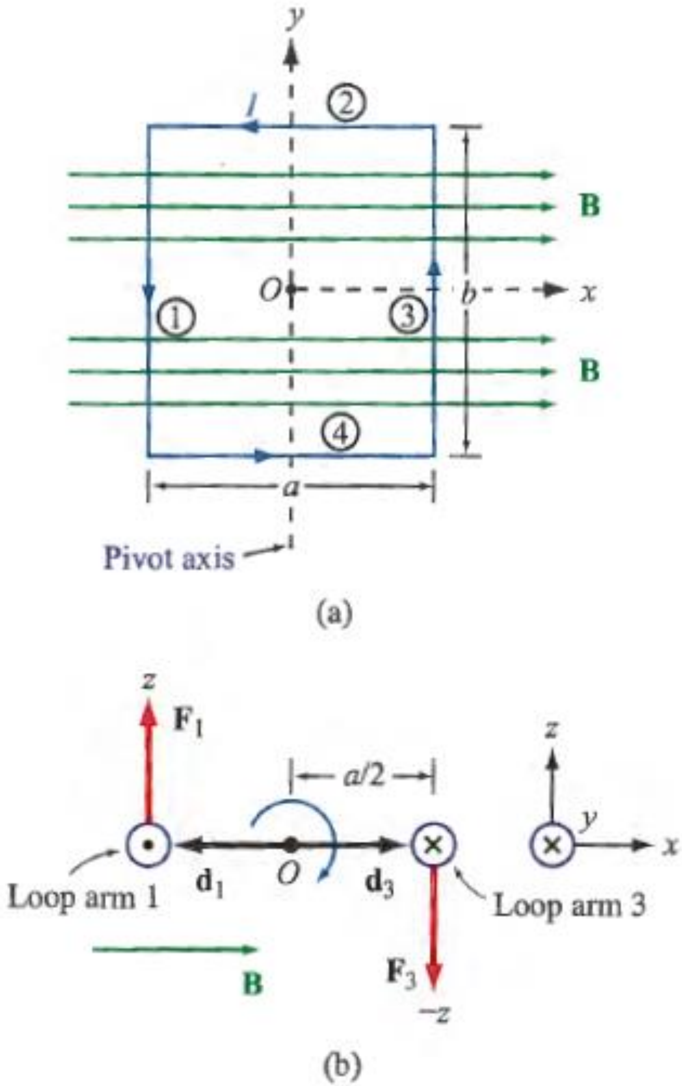
These directions are governed by the following ***right-hand rule:*** *when the thumb of the right hand points along the direction of the torque, the four fingers indicate the direction that the torque tries to rotate the body.*

Magnetic Torque on a Current Carrying Loop

We will now consider the ***magnetic torque*** exerted on a conducting loop under the influence of magnetic forces. We begin with the simple case where the magnetic field **B** is in the plane of the loop, and then we will extend the analysis to the more general case where **B** makes an angle θ with the surface normal of the loop.

Magnetic Torque on a Current Carrying Loop

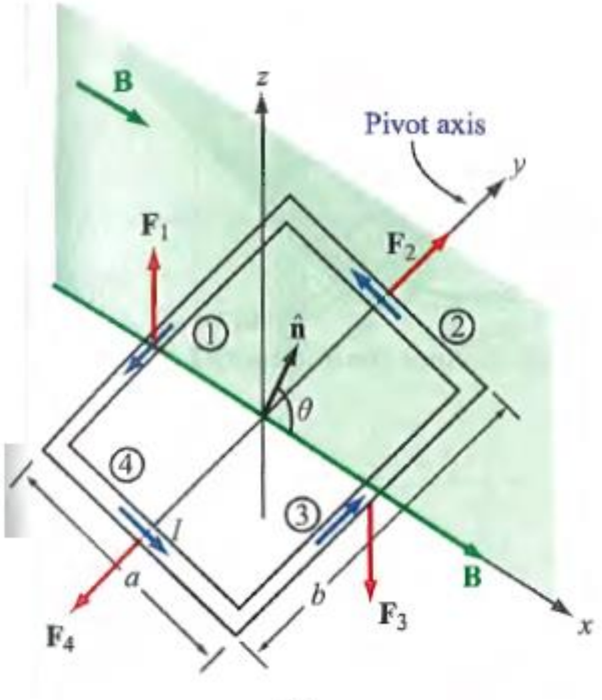
(a) Magnetic Field in the Plane of the Loop



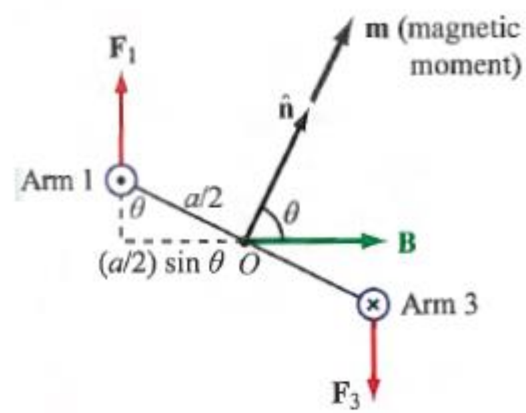
$$\begin{aligned}\mathbf{T} &= \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3 \\ &= \left(-\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (\hat{\mathbf{z}} I b B_0) + \left(\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (-\hat{\mathbf{z}} I b B_0) \\ &= \hat{\mathbf{y}} I a b B_0 = \hat{\mathbf{y}} I A B_0, \qquad (5.16)\end{aligned}$$

(b) Magnetic Field Perpendicular to the Axis of a Rectangular Loop

For the situation represented by Fig. 5-7, where $\mathbf{B} = \hat{\mathbf{x}}B_0$, the field is still perpendicular to the loop's axis of rotation, but because its direction may be at any angle θ with respect to the loop's surface normal $\hat{\mathbf{n}}$, we may now have nonzero forces on all four arms of the rectangular loop. However, forces $\mathbf{F}_2$ and $\mathbf{F}_4$ are equal in magnitude and opposite in direction and are along the rotation axis; hence, the net torque contributed by their combination is zero. The directions of the currents in arms 1 and 3 are always perpendicular to $\mathbf{B}$ regardless of the magnitude of θ. Hence, $\mathbf{F}_1$ and $\mathbf{F}_3$ have the same expressions given previously by Eqs. (5.15a and b), and for $0 \leq \theta \leq \pi/2$ their moment arms are of magnitude $(a/2)\sin\theta$, as illustrated in Fig. 5-7(b). Consequently, the magnitude of the net torque exerted by the magnetic field about the axis of rotation is the same as that given by Eq. (5.16), but modified by $\sin\theta$:



(a)



$$T = IAB_0 \sin\theta. \tag{5.17}$$

According to Eq. (5.17), the torque is maximum when the magnetic field is parallel to the plane of the loop ($\theta = 90°$) and zero when the field is perpendicular to the plane of the loop ($\theta = 0$). If the loop consists of N turns, each contributing a torque given by Eq. (5.17), then the total torque is

$$T = NIAB_0 \sin\theta. \tag{5.18}$$

The quantity NIA is called the ***magnetic moment*** m of the loop. Now, consider the vector

$$\mathbf{m} = \hat{\mathbf{n}} NIA = \hat{\mathbf{n}} m \qquad (\text{A}\cdot\text{m}^2), \tag{5.19}$$

where $\hat{\mathbf{n}}$ is the surface normal of the loop and governed by the following ***right-hand rule: when the four fingers of the right hand advance in the direction of the current I, the direction of the thumb specifies the direction of*** $\hat{\mathbf{n}}$. In terms of $\mathbf{m}$, the torque vector $\mathbf{T}$ can be written as

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N}\cdot\text{m}). \tag{5.20}$$

Magnetic Torque on a Current Carrying Loop

Exercise 5-5: A square coil of 100 turns and 0.5-m-long sides is in a region with a uniform magnetic flux density of 0.2 T. If the maximum magnetic torque exerted on the coil is 4×10^{-2} (N·m), what is the current flowing in the coil?

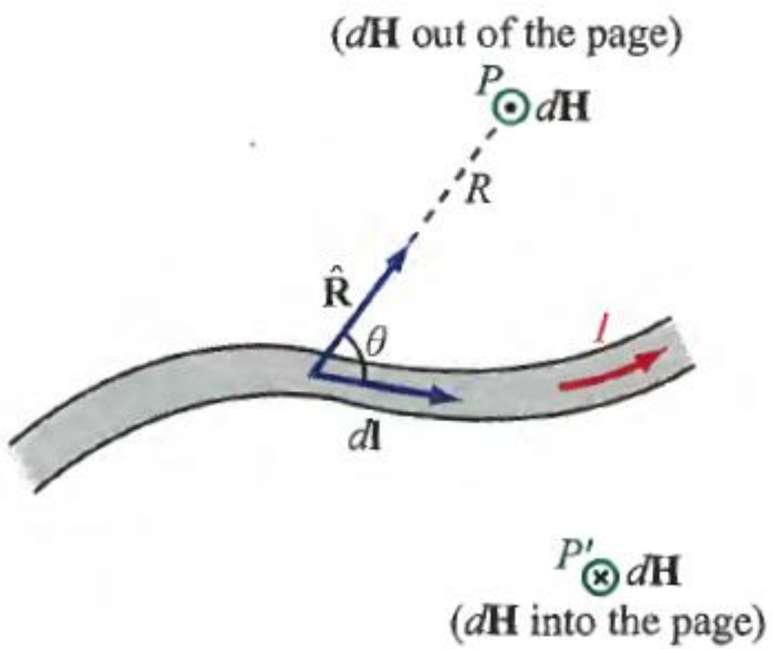
Answer: $I = 8$ mA. (See)

The Biot-Savart Law

In the preceding section, we elected to use the magnetic flux density $\mathbf{B}$ to denote the presence of a magnetic field in a given region of space. We will now work with the magnetic field intensity $\mathbf{H}$ instead. We do this in part to remind the reader that for most materials the flux and field are linearly related by $\mathbf{B} = \mu\mathbf{H}$, and therefore knowledge of one implies knowledge of the other (assuming that μ is known).

Through his experiments on the deflection of compass needles by current-carrying wires, Hans Oersted established that currents induce magnetic fields that form closed loops around the wires (see Section 1-3.3). Building upon Oersted's results, Jean Biot and Félix Savart arrived at an expression that relates the magnetic field $\mathbf{H}$ at any point in space to the current I that generates $\mathbf{H}$. The ***Biot–Savart law*** states that the differential magnetic field $d\mathbf{H}$ generated by a steady current I flowing through a differential length vector $d\mathbf{l}$ is

$$d\mathbf{H} = \frac{I}{4\pi}\,\frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}, \qquad (5.21)$$



To determine the total magnetic field **H** due to a conductor of finite size, we need to sum up the contributions due to all the current elements making up the conductor. Hence, the Biot–Savart law becomes

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \quad \text{(A/m)}, \qquad (5.22)$$

where l is the line path along which I exists.

Continued …